Exact degenerate scales in plane elasticity using complex variable methods

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A B S T R A C T

A recent work has shown that using conformal mapping can lead to exact values of the degenerate scales in plane elasticity. We elaborate on this work by introducing some algebraic tools when this conformal mapping is a rational fraction transforming the outside of the unit circle into the outside of the considered domain. Using these tools, new cases are solved including shortened hypotrochoid, arc of circle, new approximates of equilateral triangle and square or symmetric Joukowski profiles. Another method makes it possible to obtain the degenerate scales for plane elasticity from the degenerate scale for Laplace's equation for some multiply connected sets: the cases of segments on a line or of arcs of circle with a n-fold symmetry. In these last cases, the exact values of the degenerate scales are obtained when the degenerate scale for the Laplace problem is known.

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Analytical solutions
Complex variable
Elasticity
Boundary element
Plane problems
Degenerate scale

1. Introduction

The degenerate scales appear when solving single layer boundary integral equations with kernels containing a logarithmic term. This is the case for plane problems related to conduction or elasticity. Among early investigators working on Laplace's equation, we can cite Christiansen (1975); Jaswon (1963). Costabel and Dauge (1996) investigated the case of biharmonic equation. Antiplane elasticity problems are closely related to Laplace's ones and some specific cases have been considered: Joukowski profile Chen (2013), quadrilaterals Chen (2012), regular N-gon domains Kuo et al. (2013b). The case of plane elasticity has been studied in Constanda (1994); Kuhn et al. (1987); Vodička and Mantič (2004). The interest in degenerate scales has increased with the development of Boundary Element Methods, the degenerate scales causing loss of uniqueness and ill conditioning (Chen et al., 2002; Chen and Lin, 2008; Dijkstra and Mattheij, 2007) of the linear system obtained by BEM. Several methods have been developed to get over this problem (Chen et al., 2014, 2015b, 2005; Chen and Lin, 2008; Christiansen, 1982).

The asymptotic behavior of degenerate scales has been investigated for Laplace's equation Corfdir and Bonnet (2013) and for plane elasticity (Chen, 2015; Vodička, 2013). Upper bounds of degenerate scales for plane elasticity have been obtained recently Corfdir and Bonnet (2015). A first work about anisotropic elasticity has been performed by Vodička and Petrík (2015).

The exact values of degenerate scales for Laplace's Boundary Value Problems are known for many cases. They can be obtained by computing the logarithmic capacity of the domain Hayes and Kellner (1972). The name of logarithmic capacity has been given because of "the analogy with the three-dimensional Newtonian case typified by the distribution of electricity on a conductor" Hille (1962). A review of known exact values of logarithmic capacities can be found in Rumely (1989) and examples of application to Laplace's problem in Kuo et al. (2013a). In comparison, the known exact values of degenerate scales for elasticity are scarce. A review of the cases already studied can be found in Corfdir and Bonnet (2015). So, the aim of the present paper is to provide two methods of solution and the exact values of elastic degenerate scales in several application cases. The methods of solution use complex potentials. Indeed, we apply the ideas presented by Muskhelishvili (1953) to solve boundary values problems in plane elasticity using a specific complex representation (see also (England, 2003; Milne-Thomson, 1960; Sokolnikoff, 1956)). This method has been applied to numerous problems, for example the study of stress concentration due to holes and cracks (Savin, 1961; Sneddon and Lowengrub, 1969). A first application of such a method to the degenerate scale problem for elasticity was described in Chen et al. (2009a). These authors use conformal mappings $w(z)$ from the outside of the unit disk to the outside of considered domains. One feature of these conformal mappings $w(z)$ is to behave as z at ∞.

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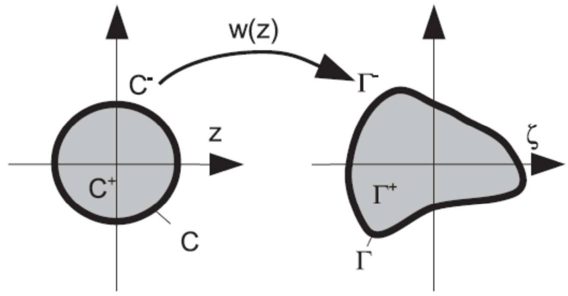

Fig. 1. Notations: conformal mapping w from C^- to Γ^-.

In this paper, a first new method is developed by extending the methodology described in the pioneering paper of Chen et al. (2009a), leading to a more systematic means to obtain the degenerate scales. The computation leads to finite algebraic linear systems. Then, the calculus can be greatly alleviated by the use of symbolic computational softwares.

A second method is developed, using the solution of the Laplace's problem to build the elastic complex potentials for two cases: sets of segments on a line or set of arcs of a circle with a n-fold symmetry axis. We show how to find the exact values of the elasticity degenerate scales in these cases when the exact value of the degenerate scale (or of the logarithmic capacity) for Laplace's problem is known.

2. The null condition at the boundary using conformal mapping and complex potentials

In this paper, the degenerate scale problem for elasticity will be studied for contours that can be described by using conformal mappings (For example, contour Γ in Fig. 1). This section presents the background on conformal mapping and complex potentials that are used to find the degenerate scales. In a first step, the requirements on the conformal mappings that are used to describe different contours are prescribed. Next, the complex potentials that allow us to obtain the solution of plane elasticity problem are recalled, these potentials being also submitted to precise requirements. Degenerate scales correspond to specific contours such that non-null potentials meet a condition of null prescribed displacement at such contours. So, the following step will be to present how this condition of null displacement at the boundary can be prescribed. These non-null potentials will be called in the following "eigenfunctions with 0 eigenvalue".

2.1. Choice of the type of conformal mapping

In the books of Muskhelishvili (1953) and Sokolnikoff (1956) the conformal mappings considered for the study of infinite domains are from the interior of the circle to the exterior of the image of the circle; more recent authors (England, 2003; Milne-Thomson, 1960) find generally more convenient to use the transformation from the outside of the circle to the outside of the image of the circle (Fig. 1). This choice is coherent with the mathematical definition of the class Σ of univalent functions Duren (1983, e.g.); it is also used for the evaluation of the logarithmic capacity and of the degenerate scale for Laplace's equation. As explained thereafter, the contours Γ that will be obtained by the conformal mappings $w(z)$ in Σ are at the degenerate scale for Laplace's equation.

We consider mapping functions defined on C^- and which can be written in the following way:

$$\zeta = Rw(z), \tag{1}$$

where R is a positive real and $\lim\limits_{z\to\infty}\frac{w(z)}{z} = 1$. Following Duren (1983), the function $w(z)$ is a univalent function of class Σ, i.e. holomorphic on C^- except for a simple pole at infinity with residue 1, and has a series expansion $w(z) = z + \sum_{n=0}^{\infty} m_n z^{-n}$.

In a first step we assume that $0 \notin \Gamma^-$; that is $w \in \Sigma*$ Duren (1983). If it is not the case, it is shown in Section 2.4 that a convenient translation allows us to transform the problem into another one that meets that condition.

It is known that the image of the unit circle by such a mapping has a logarithmic capacity equal to $\ln R$ and is at the degenerate scale for the Laplace's operator if $R = 1$ (Hayes and Kellner, 1972; Kuo et al., 2013a; Yan and Sloan, 1988). So, we will essentially compare the degenerate scales for elasticity to the degenerate scale for Laplace's problem.

2.2. Use of the elastic complex potentials

Following Muskhelishvili (1953), two elastic complex potentials Φ and Ψ can be used to obtain a displacement field solution of the plane elasticity equations in ζ plane. They are written:

$$\Phi(\zeta) = A\ln(\zeta) + C\zeta + \Phi_0(\zeta), \tag{2}$$

$$\Psi(\zeta) = -\overline{A}\ln(\zeta) + C'\zeta + \Psi_0(\zeta) \tag{3}$$

where $\overline{A}$ is the conjugate of A and functions Φ_0 and Ψ_0 are holomorphic in Γ^-. Then, the displacement components u and v are given by:

$$2G(u + iv) = \kappa\Phi - \zeta\overline{\Phi'} - \overline{\Psi} \tag{4}$$

with $\kappa = 3 - 4\nu$ for plane strain problem ($\kappa = (3-\nu)/(1+\nu)$ for plane stress problems).

Following Vodička and Mantič (2004), it is also required for the eigenfunctions with zero eigenvalue to meet the condition that the stress field induces a finite resultant force at the boundary and tends to 0 at infinity. A first consequence is that the solutions to the zero eigenvalue problem are such that $C = C' = 0$.

A second consequence of that condition impacts also the value of Φ_0 and Ψ_0 at infinity. We refer to Chen et al. (2009b) and we adopt the following values for the potentials of a concentrated force P at a point t with components P_x and P_y England (2003):

$$\Phi_P = -F\ln(z-t);$$

$$\Psi_P = \kappa\overline{F}\ln(z-t) + F\frac{\overline{t}}{z-t} \text{ with } F = -\frac{P_x + iP_y}{2\pi(\kappa+1)}. \tag{5}$$

Then, as shown in Chen et al. (2009b), the potentials at infinity can be written in the form (2,3) with $C = C' = 0$ and Φ_0 and Ψ_0 tend to zero when z tends to infinity. This form of Φ_0 and Ψ_0 depends on the choice of the complex potentials Φ_P and Ψ_P related to the concentrated force and is true only for the choice given by (5). It is important to notice that this form of potential for a point force corresponds to an expression of the Green's tensor for plane elasticity that is not the standard Green's tensor (Kelvin's tensor). In addition, modifying the choice of the potential that is used to describe the concentrated forces leads to another value of the degenerate scale. A scaling procedure to convert the degenerate scale for one choice of potential to the degenerate scale for another one is reported in Vodička and Mantič (2004, (3.4)). A consequence is that all degenerate scales obtained in the following must be multiplied by the factor $e^{\frac{1}{2\kappa}}$ to recover the degenerate scales corresponding to the usual Kelvin's tensor.

2.3. The boundary equation for eigenfunctions with 0 eigenvalue

We are looking for non trivial solutions with a null displacement on Γ: then the potentials $\Phi(\zeta)$, $\Psi(\zeta)$ must satisfy a boundary equation that is taken as the conjugate of the condition that comes directly from the definition of the displacement field (4):

$$\kappa\overline{\Phi}(\zeta) - \overline{\zeta}\Phi'(\zeta) - \Psi(\zeta) = 0 \ \forall\zeta \in \Gamma. \tag{6}$$

This condition is now transformed by the mapping into a condition in the z plane. Therefore, we consider now the two functions $\phi(z)$, $\psi(z)$ defined on $C^- \cup C$:

$$\phi(z) = \Phi(Rw(z));\ \psi(z) = \Psi(Rw(z)). \tag{7}$$

These two functions are multivalued (due to the logarithmic terms in ([2,3])) and holomorphic in C^-. Substituting (7) and $\phi'(z) = Rw'(z)\Phi'(Rw(z))$ into (6), we get the following boundary condition (in the z plane) on the unit circle:

$$\kappa\overline{\phi}(z) - \frac{\overline{w}(z)}{w'(z)}\phi'(z) - \psi(z) = 0 \ \forall z \in C. \tag{8}$$

It is now possible to introduce into this boundary condition the expression of ϕ and ψ containing the logarithmic terms.

We have:

$$\phi(z) = \Phi(Rw(z)) = A\ln(Rw(z)) + \phi_1(z) \text{ with } \phi_1(z) = \Phi_0(Rw(z)) \tag{9}$$

with a similar relation for ψ. ϕ_1 and ψ_1 are univalued holomorphic functions in C^- that are continuous in $C^- \cup C$.

These expressions contain the multivalued logarithmic term that can be written as:

$$\ln(Rw(z)) = \ln(R) + \ln(z) + \underbrace{(\ln(w(z)) - \ln(z))}_{\phi_2(z)}. \tag{10}$$

We intend to show that ϕ_2 is univalued in C^- and tends to zero when $|z|$ tends to infinity. This is proved by using an argument developed in Muskhelishvili (1953). For any closed circuit in C^- not passing around C, the function ϕ_2 reverts to its former value. However, it is possible to show also that property for a closed circuit passing around C. Indeed, the derivative ϕ_2' is $O(1/z^2)$, then the increase along a circle centered at the origin with radius T is equal to the integral of ϕ_2' along this circle and is $O(1/T)$. Therefore, it is equal to zero since it does not depend on $T > 1$. As a consequence, ϕ_2 can be chosen as a univalued holomorphic function in C^- tending to zero when $|z| \to \infty$, since $\frac{w}{z}$ tends to 1 for large z.

Using the previous decompositions of ϕ, ψ, $\ln(Rw(z))$ leads to:

$$\phi(z) = A\ln(R) + A\ln(z) + \phi_0(z) \tag{11}$$

$$\text{with } \phi_0(z) = A\phi_1(z) + \phi_2(z); \tag{12}$$

$$\psi(z) = -\overline{A}\kappa\ln(R) - \overline{A}\kappa\ln(z) + \psi_0(z) \tag{13}$$

$$\text{with } \psi_0(z) = -\overline{A}\kappa\psi_1(z) + \psi_2(z), \tag{14}$$

where the functions ϕ_0 and ψ_0 are univalued, holomorphic in C^- and tend to zero when $z \to \infty$.

Inserting (11) and (13) into the boundary Eq. (8), we get the following equation (Chen et al., 2009a):

$$2\kappa\overline{A}\ln(R) + \kappa\overline{\phi_0}(z) - \tilde{f}(z)\phi_0' - A\tilde{f}(z)\frac{1}{z} - \psi_0(z) = 0 \ \forall z \in C, \tag{15}$$

where $\tilde{f} = \frac{\overline{w}(z)}{w'(z)}$.

The displacement given by the conjugate of the left hand term of (15) must also be continuous in $C^- \cup C$. The values of R for which it is possible to find non-null values of A, ϕ_0 and ψ_0 meeting (15) are the degenerate scales for Γ.

Chen et al. (2009b) have found a few solutions of this equation and obtained the value of the degenerate scales in the related cases.

2.4. *Remark: effect of adding a constant term to* w(*z*)

If we translate the contour Γ by adding a constant w_0 to $w(z)$, its degenerate scale does not change. As a consequence, the solutions of Eq. (15) and of its transformed by a translation must be the same.

Indeed, if we substitute $w(z) + w_0$ to $w(z)$ in (15), we get:

$$\begin{aligned} &2\kappa\overline{A}\ln(R) + \kappa\overline{\phi}_0(z) - \tilde{f}(z)\phi_0'(z) - \frac{\overline{w_0}}{w'(z)}\phi_0'(z) - A\tilde{f}(z)\frac{1}{z} \\ &-A\frac{\overline{w_0}}{w'(z)}\frac{1}{z} - \psi_0(z) = 2\kappa\overline{A}\ln(R) + \kappa\overline{\phi}_0(z) - \tilde{f}(z)\phi_0'(z) \\ &-A\tilde{f}(z)\frac{1}{z} - \left(\frac{\overline{w_0}}{w'(z)}\phi_0'(z) - A\frac{\overline{w_0}}{w'(z)}\frac{1}{z} - \psi_0(z)\right) = 0. \end{aligned} \tag{16}$$

We first notice that if the set $(A, \phi_0(z), \psi_0(z))$ is non-null and satisfies (15), then $A \neq 0$ (otherwise any value of R would be a degenerate scale). We deduce that if $(A, \phi_0(z), \psi_0(z))$ is a non-null solution of (15) for the mapping $(w(z) + w_0)$ at the scale R, then $(A, \phi_0(z), \frac{\overline{w_0}}{w'(z)}\phi_0'(z) - A\frac{\overline{w_0}}{w'(z)}\frac{1}{z} - \psi_0(z)$ is a non-null solution of (15) for the mapping $w(z)$ at the same scale R. The derivation of (15) used the fact that the mapping must be in Σ^*. In fact it is only needed that there exists at least one w_0 such that the condition $w(z) + w_0 \in \Sigma^*$ is fulfilled and this is always possible. In addition, it proves finally that Γ and its transform by a translation lead to equations in A, ϕ_0 and ψ_0 that provide the same values of degenerate scales.

3. A method for the case where the conformal mapping is a rational fraction

As explained previously, Eq. (15) can be used to find degenerate scales, on the assumption that $w \in \Sigma$. However, it can be noticed that all the examples for degenerate scales solved in Chen et al. (2009a) use conformal mappings that are rational fractions. In addition, this kind of conformal mapping is also the only one for which a method of solution is developed in classical books (England, 2003; Muskhelishvili, 1953).

It is therefore of prime importance to develop more precisely this case. This is the subject of this section. In a first step, it will be shown that an alternative equation can be used to obtain the degenerate scales. Next, this equation will be transformed into a linear system of equations that will be used to provide the degenerate scales.

3.1. *A new expression of the boundary equation if* w(z) *is a rational fraction*

We suppose that $w(z)$ is continuous on $C \cup C^-$, so $w(z)$ has no pole on C. In Section 3 and the two following ones, it is assumed that $w'(z) \neq 0, z \in C$ (we already know that $w'(z) \neq 0, z \in C^-$ since w is a conformal mapping).

As $w(z) = O(z)$ when $z \to \infty$ and as we choose a null constant term as explained previously, we can write:

$$w(z) = z + \sum_{j=1}^{l_0}\frac{m_j}{z^j} + \sum_{i=1}^{n}\sum_{j=1}^{l_i}\frac{m_{i,j}}{(z - y_i)^j}. \tag{17}$$

The poles $0, y_i\ i \in \{1, \ldots, n\}$ are all in C^+.

We define $\tilde{w}(z)$ as:

$$\tilde{w}(z) = \frac{1}{z} + \sum_{j=1}^{l_0}\frac{\overline{m_j}}{z^{-j}} + \sum_{i=1}^{n}\sum_{j=1}^{l_i}\frac{\overline{m_{i,j}}}{(1/z - \overline{y_i})^j}\ z \in C^-, \tag{18}$$

that corresponds to $\tilde{w}(z) = \overline{w(z)}$, $z \in C$.

The function $\tilde{f}(z)$ can be written:

$$\tilde{f}(z) = \frac{\tilde{w}(z)}{w'(z)}. \tag{19}$$

The function $\widetilde{f}(z)$ is also a rational fraction. As the derivative $w'(z)$ has all its zeros in C^+, the poles of $\widetilde{f}(z)$ in C^- are $(z_i = 1/\overline{y_i}, \infty)$, the inverses of the conjugates of the poles of $w(z)$ with the same multiplicity. This function can be written in the following way:

$$\widetilde{f}(z) = (b_0 + \sum_{j=1}^{l_0} b_i z^i) + \sum_{i=1}^{n}\sum_{j=1}^{l_i} \frac{b_{i,j}}{(z-z_i)^j} + f_1(z), \tag{20}$$

where $f_1(z)$ is a rational fraction that is holomorphic in C^-, continuous in $C^- \cup C$, with all its poles in C^+ and vanishing when $|z| = \infty$.

We assume that $\phi_0'(z)$ is holomorphic in C^-, continuous in $C^- \cup C$ and tends to zero when $z \to \infty$. Under this condition, the product $\widetilde{f}(z)\phi_0'(z)$ can be written as:

$$\widetilde{f}(z)\phi_0'(z) = \Omega(z) = \underbrace{(B_0 + \sum_{j=1}^{l_0} B_i z^i)}_{=P(z)} + \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{l_i} \frac{B_{i,j}}{(z-z_i)^j}}_{=F(z)} + f_2(z) \tag{21}$$

with $f_2(z)$ a function holomorphic in C^-, continuous in $C^- \cup C$ and vanishing when $|z| = \infty$.

Similarly, we can write:

$$\widetilde{f}(z)\frac{1}{z} = \omega(z) = \underbrace{(C_0 + \sum_{j=1}^{l_0} C_i z^i)}_{=Q(z)} + \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{l_i} \frac{C_{i,j}}{(z-z_i)^j}}_{=G(z)} + f_3(z), \tag{22}$$

where $f_3(z)$ is a function that is holomorphic in C^-, continuous in $C^- \cup C$ and vanishing when $|z| = \infty$.

Substituting relations (21) and (22) into (15) leads to:

$$\begin{aligned} &2\kappa\overline{A}\ln(R) + \kappa\overline{\phi_0(z)} - P(z) - F(z) - f_2(z) \\ &-AQ(z) - AG(z) - Af_3(z) - \psi_0(z) = 0 \quad \forall z \in C. \end{aligned} \tag{23}$$

3.2. Simplified form of the boundary condition using Cauchy integrals

The form of the boundary condition (23) can be simplified using the properties of Cauchy integrals. Applying the operator $f \to \frac{1}{2\pi i}\int_C \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z}$ to (23), we use the classical results on Cauchy integrals (Muskhelishvili, 1953) that are recalled in Appendix A.

The image of $2\kappa\overline{A}$ is null by (A.2), the image of $\kappa\overline{\phi_0(z)}$ is null by (A.3), the images of P, F, Q, G are null by (A.4). Finally we get by (A.1):

$$f_2(z) + Af_3(z) + \psi_0(z) = 0 \ \forall z \in C^-. \tag{24}$$

Adding Eqs. (23) to (24) leads to:

$$2\kappa\overline{A}\ln(R) + \kappa\overline{\phi_0(z)} - P(z) - F(z) - AQ(z) - AG(z) = 0 \ \forall z \in C. \tag{25}$$

Considering the conjugate of (25), we find:

$$2\kappa A\ln(R) + \kappa\phi_0(z) - \overline{P(z)} - \overline{F(z)} - \overline{A}\,\overline{Q(z)} - \overline{A}\,\overline{G(z)} = 0 \ \forall z \in C. \tag{26}$$

We define:

$$\widetilde{P}(z) = \sum_{j=1}^{l_0} \overline{B_j}\frac{1}{z^j}, \tag{27}$$

with the property that:

$$\overline{P(z)} = \overline{B_0} + \widetilde{P}(z) \ \forall z \in C. \tag{28}$$

In the same way, $\widetilde{Q}(z)$ is defined by the equality:

$$\overline{Q(z)} = \overline{C_0} + \widetilde{Q}(z) \ \forall z \in C, \tag{29}$$

and $\widetilde{F_0}$ is defined by:

$$\begin{aligned} \widetilde{F_0}(z) &= \sum_{i=1}^{n}\sum_{j=1}^{l_i} \overline{\left(\frac{B_{i,j}}{(z-z_i)^j}\right)} = \sum_{i=1}^{n}\sum_{j=1}^{l_i} \frac{\overline{B_{i,j}}}{(1/z - 1/y_i)^j} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{l_i} \frac{\overline{B_{i,j}} z^j y_i^j}{(y_i - z)^j} \ \forall z \in C. \end{aligned} \tag{30}$$

It must be noticed that:

$$\frac{z^j y_i^j}{(y_i - z)^j} = \left(y_i\left(-1 - \frac{y_i}{z - y_i}\right)\right)^j = (-1)^j y_i^j \sum_{k=0}^{j} \binom{j}{k} \frac{y_i^k}{(z - y_i)^k}, \tag{31}$$

so the above expression is the sum of a holomorphic function on C^- vanishing at infinity and of the constant $(-y_i)^j$. We define now:

$$\widetilde{F}(z) = \widetilde{F_0}(z) - F_0 \tag{32}$$

with:

$$F_0 = \sum_{i=1}^{n}\sum_{j=1}^{l_i} \overline{B_{i,j}}(-y_i)^j. \tag{33}$$

And finally we have:

$$\overline{F(z)} = F_0 + \widetilde{F}(z) \ \forall z \in C. \tag{34}$$

$\widetilde{G}$, G_0 are defined in the same way with:

$$\overline{G(z)} = G_0 + \widetilde{G}(z) \ \forall z \in C. \tag{35}$$

Substituting (28), (29), (34) and (35) into Eq. (26) and then applying $f \to \frac{1}{2\pi i}\int_C \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z}$ leads to:

$$-\kappa\phi_0(z) + \widetilde{P}(z) + \widetilde{F}(z) + \overline{A}\,\widetilde{Q}(z) + \overline{A}\,\widetilde{G}(z) = 0 \ \forall z \in C^-. \tag{36}$$

Eq. (36) shows that ϕ_0 is a rational fraction. Such a rational fraction is holomorphic in the whole plane except at poles. It has a Laurent series in the neighborhood of ∞: $\phi_0 = \sum_{i=1}^{l_0} a_i/z^i$. Then we deduce the following set of l_0 equations:

$$\kappa a_i - \overline{B_i} - \overline{A}\,\overline{C_i} = 0 \quad \forall i \in \{1, \ldots, l_0\}. \tag{37}$$

All the functions in (36) are rational fractions and can be differentiated as suggested in Muskhelishvili (1953), leading to $(\sum_{i=1}^{n} l_j)$ relations:

$$\begin{aligned} &-\kappa\phi_0^{(i)}(z_j) + \widetilde{P}^{(i)}(z_j) + \widetilde{F}^{(i)}(z_j) + \overline{A}\,\widetilde{Q}^{(i)}(z_j) + \overline{A}\,\widetilde{G}^{(i)}(z_j) \\ &= 0 \ \forall i \in \{1, \ldots, l_j\}. \end{aligned} \tag{38}$$

Adding (36) and (26) leads to:

$$2\kappa A\ln(R) - \overline{B_0} - F_0 - \overline{A}\,\overline{C_0} - \overline{A}\,G_0 = 0. \tag{39}$$

3.3. Computation of the degenerate scales by using the determinant of homogeneous systems

The set of $(\sum_{i=0}^{n} l_i + 1)$ Eqs. (37)–(39) in $(\sum_{i=0}^{n} l_i + 1)$ unknown variables $(A, a_i, \phi_0^{(i)}(z_j))$ constitute a squared homogeneous linear system. Splitting each equation in real and imaginary parts, we get $2(\sum_{i=0}^{n} l_i + 1)$ linear equations and $2(\sum_{i=0}^{n} l_i + 1)$ real unknown variables $(\mathrm{Re}(A), \mathrm{Im}(A), \mathrm{Re}(a_i), \mathrm{Im}(a_i), \mathrm{Re}(\phi_0^{(i)}(z_j)), \mathrm{Im}(\phi_0^{(i)}(z_j))$ that constitute again a squared homogeneous linear system. There are non trivial solutions of this system of equations iff its determinant is null.

This determinant is a polynomial of degree 2 in $\rho = \ln R$ that leads to two values of the degenerate scale, as known in the case of plane elasticity. In Section 4, we will evaluate completely the determinant of this system in some specific cases. This constitutes the first method used thereafter.

3.4. An alternative method

A second method can be described as follows. Since ϕ_0 is a rational fraction, it is possible to write:

$$\phi_0(z) = \sum_{j=1}^{l_0} g_i z^{-i} + \sum_{i=1}^{n} \sum_{g=1}^{l_i} \frac{g_{i,j}}{(z - z_i)^j} \tag{40}$$

$$\widetilde{\phi_0}(z) = \sum_{j=1}^{l_0} \overline{g_i} z^i + \sum_{i=1}^{n} \sum_{g=1}^{l_i} \frac{\overline{g_{i,j}}}{(1/z - 1/\overline{z_i})^j}. \tag{41}$$

We consider:

$$2\kappa \overline{A} \ln(R) + \kappa \widetilde{\phi_0}(z) - \widetilde{f}(z)\left(\phi_0'(z) + A/z\right). \tag{42}$$

It is possible to find ψ a holomorphic function in C^-, vanishing at ∞, satisfying (15), if the above rational fraction has no constant term, no terms z^i, no terms $1/(z - y_i)^j$. These conditions give $(\sum_{i=0}^{n} l_i + 1)$ equations with the $(\sum_{i=0}^{n} l_i + 1)$ unknown variables g_i, $g_{i,j}$. This is an equivalent system to the one described in the first method but the unknown variables are not the same. However, the algebra has been found simpler than the one obtained in Section 3.2 in several cases. The degenerate scales are obtained again by computing explicitly the determinant of this system of equations, as explained in Section 3.3. This alternative method will be used in Section 5.1 and in Example 6.5.

3.5. Remark on the effect of changing κ into $-\kappa$

It is noteworthy that the system of equations described above is invariant when changing κ into $-\kappa$, as it will be shown now.

As explained before, the determinant of the system is a polynomial of degree 2 in $\rho = \ln R$. We can write it as:

$$S(\kappa, b_i, \overline{b_i}, b_{i,j}, \overline{b_{i,j}})\rho^2 + T(\kappa, b_i, \overline{b_i}, b_{i,j}, \overline{b_{i,j}})\rho + U(\kappa, b_i, \overline{b_i}, b_{i,j}, \overline{b_{i,j}}). \tag{43}$$

The functions (S, T, U) are polynomials. We consider Eq. (15) and we replace A by $\widehat{A} = A/i$,ϕ_0 by $\widehat{\phi_0} = \phi_0/i$, ψ_0 by $\widehat{\psi_0} = \psi_0/i$. After dividing by i, we get the new equation:

$$-2\kappa \overline{\widehat{A}}\rho - \kappa \overline{\widehat{\phi}}_0(z) - \widetilde{f}(z)\widehat{\phi}_0' - \widehat{A}\widetilde{f}(z)\frac{1}{z} - \widehat{\psi}_0(z) = 0 \; \forall z \in C. \tag{44}$$

It is the same equation as (15) if we replace $-\kappa$ by κ. The corresponding equation for ρ is:

$$S(-\kappa, b_i, \overline{b_i}, b_{i,j}, \overline{b_{i,j}})\rho^2 + T(-\kappa, b_i, \overline{b_i}, b_{i,j}, \overline{b_{i,j}})\rho + U(-\kappa, b_i, \overline{b_i}, b_{i,j}, \overline{b_{i,j}}). \tag{45}$$

Then, if ρ is a solution of (43) it is also a solution of (45). So, if we change the sign of κ in the algebraic expression of $\rho = \ln(R_c)$, we get also the expression of the logarithm of one of the critical scales. It must be noticed that this property is true only for the present choice of the elastic potential and of the potentials related to the concentrated forces (see Section 2.2).

4. Case of a conformal mapping $w(z) = z + P(z^{-1})$

4.1. General case of P

We assume that P is a polynomial of degree $n \geq 2$ of $\frac{1}{z}$: the case $n = 1$ is the case of an ellipse and is solved in details in Chen et al. (2009b).

$$P = \sum_{i=1}^{n} m_i \frac{1}{z^i}. \tag{46}$$

Eq. (20) can then be written:

$$\widetilde{f}(z) = b_n z^n + b_{n-1} z^{n-1} \cdots + b_0 + f_1(z) \tag{47}$$

with $f_1(z)$ holomorphic in C^-, continuous in $C^- \cup C$ and vanishing at ∞ (we still assume that $w'(z) \neq 0$; $z \in C$).

We write:

$$\phi_0 = \sum_{i=1}^{n} a_i z^{-i}. \tag{48}$$

Eqs. (37) and (39) can be written:

$$2\kappa \ln(R)\overline{A} + B_0 - Ab_1 = 0 \text{ for } k = 0; \tag{49}$$

$$\kappa \overline{a}_k + B_k - Ab_{k+1} = 0 \text{ for } (n-1) > k > 0; \tag{50}$$

$$\kappa \overline{a}_{(n-1)} - Ab_n = 0 \text{ for } k = n - 1; \tag{51}$$

$$\kappa \overline{a}_j = 0 \text{ for } k > n - 1; \tag{52}$$

with

$$B_k = \sum_{j-i=k+1} i a_i b_j. \tag{53}$$

Substituting the value of B_k in (49)–(52), we get a system of n equations in the unknown variables $A, a_1, \ldots, a_k, \ldots, a_{n-1}$ and their conjugates:

$$2\kappa \ln(R)\overline{A} + a_1 b_2 + 2a_2 b_3 + \cdots + (n-1)a_{n-1} b_n - Ab_1 = 0; \tag{54}$$

$$\kappa \overline{a}_1 + a_1 b_3 + 2a_2 b_4 + \cdots + (n-2)a_{n-2} b_n - Ab_2 = 0; \tag{55}$$

$$\kappa \overline{a}_{k-1} + \sum_{j-i=k+1} i a_i b_j - Ab_k = 0; \tag{56}$$

$$\kappa \overline{a}_{(n-2)} + a_1 b_n - Ab_{n-1} = 0; \tag{57}$$

$$\kappa \overline{a}_{(n-1)} - Ab_n = 0. \tag{58}$$

Each of the above equations can be split into real and imaginary parts. Then we get a system of $2n$ equations with the unknown variables $\mathrm{Re}(A)$, $\mathrm{Im}(A)$, $\mathrm{Re}(a_i)$, $\mathrm{Im}(a_i)$ for $i = 1 \ldots (n-1)$. This system has the following determinant with $\rho = \ln(R)$, $b_i = c_i + id_i$:

$$\begin{vmatrix}
2\kappa\rho - c_1 & d_1 & c_2 & -d_2 & \cdots & (n-2)c_{n-1} & -(n-2)d_{n-1} & (n-1)c_n & -(n-1)d_n \\
-d_1 & -2\kappa\rho - c_1 & d_2 & c_2 & \cdots & (n-2)d_{n-1} & (n-2)c_{n-1} & (n-1)d_n & (n-1)c_n \\
-c_2 & d_2 & \kappa + c_3 & -d_3 & \cdots & (n-2)c_n & -(n-2)d_n & 0 & 0 \\
-d_2 & -c_2 & d_3 & -\kappa + c_3 & \cdots & (n-2)d_n & (n-2)c_n & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-c_{n-1} & d_{n-1} & c_n & -d_n & \cdots & \kappa & 0 & 0 & 0 \\
-d_{n-1} & -c_{n-1} & d_n & c_n & \cdots & 0 & -\kappa & 0 & 0 \\
-c_n & d_n & 0 & 0 & \cdots & 0 & 0 & \kappa & 0 \\
-d_n & -c_n & 0 & 0 & \cdots & 0 & 0 & 0 & -\kappa
\end{vmatrix}. \tag{59}$$

With the help of a computer algebra software, the coefficients b_i and the determinant can be easily computed. When equating the determinant to zero, we get an equation of the second degree in $\rho = \ln(R)$ with two complex solutions taking into account the multiplicity. In fact the solutions are always real since they determine the two degenerate scales. In the following, we will study different particular cases.

4.2. *Case where the* b_i *are real numbers*

In that case the determinant can be written as:

$$\begin{vmatrix} 2\kappa\rho - c_1 & 0 & c_2 & 0 & \dots & (n-2)c_{n-1} & 0 & (n-1)c_n & 0 \\ 0 & -2\kappa\rho - c_1 & 0 & c_2 & \dots & 0 & (n-2)c_{n-1} & 0 & (n-1)c_n \\ -c_2 & 0 & \kappa + c_3 & 0 & \dots & (n-2)c_n & 0 & 0 & 0 \\ 0 & -c_2 & 0 & -\kappa + c_3 & \dots & 0 & (n-2)c_n & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -c_{n-1} & 0 & c_n & 0 & \dots & \kappa & 0 & 0 & 0 \\ 0 & -c_{n-1} & 0 & c_n & \dots & 0 & -\kappa & 0 & 0 \\ -c_n & 0 & 0 & 0 & \dots & 0 & 0 & \kappa & 0 \\ 0 & -c_n & 0 & 0 & \dots & 0 & 0 & 0 & -\kappa \end{vmatrix}. \quad (60)$$

It can be arranged by moving upwards all uneven lines, and then moving leftwards all uneven columns and finally we get this new expression of the determinant:

$$\begin{vmatrix} M^+ & 0_n \\ 0_n & M^- \end{vmatrix}, \quad (61)$$

where 0_n is the $n \times n$ null matrix and M^+ and M^- are written below:

$$M^+ = \begin{pmatrix} 2\kappa\rho - c_1 & c_2 & \dots & (n-2)c_{n-1} & (n-1)c_n \\ -c_2 & \kappa + c_3 & \dots & (n-2)c_n & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -c_{n-1} & c_n & \dots & \kappa & 0 \\ -c_n & 0 & \dots & 0 & \kappa \end{pmatrix}; \quad (62)$$

$$M^- = \begin{pmatrix} -2\kappa\rho - c_1 & c_2 & \dots & (n-2)c_{n-1} & (n-1)c_n \\ -c_2 & -\kappa + c_3 & \dots & (n-2)c_n & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -c_{n-1} & c_n & \dots & -\kappa & 0 \\ -c_n & 0 & \dots & 0 & -\kappa \end{pmatrix}. \quad (63)$$

Finally, this determinant can be expanded as the product of two determinants $D^+ = |M^+|$ and $D^- = |M^-|$. Both are one degree polynomials in $\rho = \ln(R)$.

These results will be used in some particular cases described in the following subsection.

4.3. *Application to the case of a n-fold axis of symmetry:* $w(z) = z + zP(z^{-n})$

We assume that P is a polynomial in the variable z^{-n} with no constant term. We assume in the following that all coefficients of P are real. This mapping can be used to find the degenerate scales corresponding to several interesting contours.

4.3.1. *P is of degree 1*

This allows to describe the contours such as hypotrochoid that have been considered by several authors (Ivanov and Trubetskov, 1995; Muskhelishvili, 1953; Zimmerman, 1986) which provided solutions to elasticity problems for such boundaries.

We consider the following mapping:

$$w(z) = z + \frac{m}{z^{n-1}} \quad (64)$$

and we assume that $m \in \mathbb{R}^+$. We restrict ourselves to the case $m < \frac{1}{n-1}$ for having a conformal mapping such that $w'(z)$ has no zeros on C. The contour Γ is a shortened hypotrochoid.

From Eq. (20), the corresponding function $\widetilde{f}(z)$ is:

$$\widetilde{f}(z) = m.z^{n-1} + (m^2(n-1) + 1)\frac{z^{n-1}}{z^n - m(n-1)}. \quad (65)$$

The last term has all its poles in C^+ if $m < 1/(n-1)$. All the coefficients c_i are real and we have:

$$c_{n-1} = m; c_i = 0, i = 1\dots(n-2). \quad (66)$$

We assume $n \geq 3$ and we write (62) for the special case of a n-fold axis of symmetry:

$$D^+ = \begin{array}{c c} & \begin{array}{c c c} \textit{column } 1 & & (n-1) \end{array} \\ \begin{array}{c} \textit{row } 1 \\ \\ \\ n-1 \\ \end{array} & \begin{vmatrix} 2\kappa\rho & \dots & (n-2)m \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & & 0 \\ -m & \dots & \kappa \end{vmatrix} \end{array}. \quad (67)$$

Adding to the first column the last one multiplied by m/κ leads to:

$$D^+ = \begin{vmatrix} 2\kappa\rho + \frac{(n-2)m^2}{\kappa} & \dots & (n-2)m \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & & 0 \\ 0 & \dots & \kappa \end{vmatrix}. \quad (68)$$

The expansion of the determinant along its first column gives the condition for a critical scale $2\kappa\rho + (n-2)m^2/\kappa = 0$. We deduce one value of the critical scale. It can be checked that the critical value deduced from D^- is the same; but this is also a consequence of the fact that there is only one (double) critical scale for boundaries with an axial symmetry of order ≥ 3 (Vodička and Mantič, 2004). We finally write the single critical scale:

$$\ln(R_c) = -\frac{(n-2)m^2}{2\kappa^2}. \quad (69)$$

When $m \to 0$ or when $\kappa \to \infty$ the degenerate scales $R_c \to 1$; that means that both degenerate scales tend to the degenerate scale for Laplace's equation.

To end this case, it is possible to recover the degenerate scales for two special cases where the hypotrochoid approximates an equilateral triangle or a square.

The equilateral triangle is approximated by $w(z) = z + \frac{1}{3z^2}$ corresponding to $n = 3; m = 1/3$. Substituting these values into formula (69) gives $\ln(R_c) = -1/(18\kappa^2)$ (see curve T1 on Fig. 2). The square is approximated by $w(z) = z + \frac{1}{6z^3}$ corresponding to $n = 4; m = 1/6$. From (69), we get: $\ln(R_c) = -1/(36\kappa^2)$ (see curve S1 on Fig. 3). These two results that have been found previously by Chen et al. (2009a) are two examples at the first order of the more general case presented thereafter in this subsection.

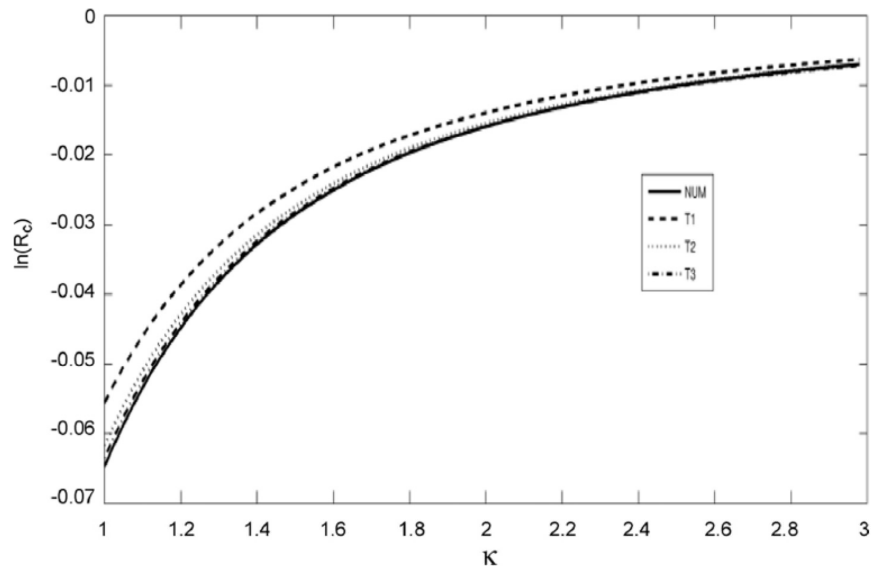

Fig. 2. Comparison of $\ln(R_c)$ as a function of κ for the approximates (T1, T2, T3), of an equilateral triangle and for the numerical computation (NUM) (in the case of a triangle at the degenerate scale for Laplace's equation).

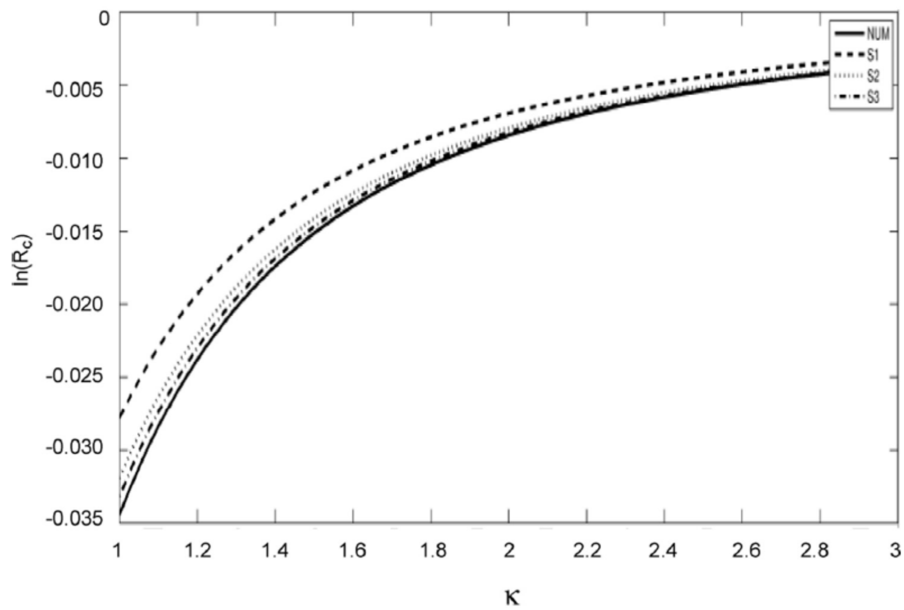

Fig. 3. Comparison of $\ln(R_c)$ as a function of κ for the approximates (S1, S2, S3), of a square and for the numerical computation (NUM) (in the case of a square at the degenerate scale for Laplace's equation).

4.3.2. P is of degree 2

We consider now the mapping corresponding to:

$$w(z) = z + \frac{m_1}{z^{n-1}} + \frac{m_2}{z^{2n-1}}. \tag{70}$$

We assume that m_1 and m_2 are real numbers. This type of conformal mapping has been considered by Ekneligoda and Zimmerman (2006, 2008) to model holes having n-fold symmetry.

The next step is to compute the rational fraction $\widetilde{f}(z)$, that is easily achieved by computing w' and $\widetilde{w}$. Then, this rational fraction is put into the form $\widetilde{f}(z) = \frac{N(z)}{D(z)}$ where N and D are polynomials. The polynomial quotient $Q(z)$ of $N(z)$ by $D(z)$ is given by $N(z) = Q(z).D(z) + R(z)$ with the degree of $R(z)$ inferior to the one of $D(z)$. This leads to:

$$\widetilde{f}(z) = Q(z) + f_1(z) \tag{71}$$

where $f_1(z) = \frac{R(z)}{D(z)}$ is a function that tends to zero at infinity.

With the previous expression of the mapping $w(z)$, Eq. (47) that comes from that process writes out:

$$\widetilde{f}(z) = m_2 z^{2n-1} + m_1(1 + (n-1)m_2)z^{n-1} + f_1(z). \tag{72}$$

If $n = 3$, the determinant D^+ can be written as:

$$D^+ = \begin{vmatrix} 2\kappa\rho & c_2 & 0 & 0 & 4c_5 \\ -c_2 & \kappa & 0 & nc_5 & 0 \\ 0 & 0 & 2c_3+\kappa & 0 & 0 \\ 0 & c_5 & 0 & \kappa & 0 \\ -c_5 & 0 & 0 & 0 & \kappa \end{vmatrix}. \tag{73}$$

We expand the determinant along the third row (or column). We get:

$$D^{+} = (2c_5 + \kappa)\begin{vmatrix} 2\kappa\rho & c_2 & 0 & 4c_5 \\ -c_2 & \kappa & nc_5 & 0 \\ 0 & c_5 & \kappa & 0 \\ -c_5 & 0 & 0 & \kappa \end{vmatrix}. \tag{74}$$

If $n > 3$, the determinant D^{+} can be written as:

$$\begin{array}{r|ccccccc|} & \mathit{column}\ 1 & & (n-1) & n & (n+1) & & (2n-1) \\ \mathit{row}\ 1 & 2\kappa\rho & \cdots & (n-2)c_{n-1} & 0 & 0 & & (2n-2)c_{2n-1} \\ & 0 & \ddots & \vdots & 0 & \vdots & ⋰ & 0 \\ n-1 & -c_{n-1} & \cdots & \kappa & 0 & nc_{2n-1} & & 0 \\ n & 0 & \cdots & 0 & \kappa + (n-1)c_{2n-1} & 0 & & 0 \\ n+1 & 0 & & (n-2)c_{2n-1} & 0 & \kappa & & 0 \\ & 0 & ⋰ & \vdots & \vdots & \vdots & \ddots & 0 \\ 2n-1 & -c_{2n-1} & \cdots & 0 & 0 & 0 & \cdots & \kappa \end{array}. \tag{75}$$

This expression is the same as the previous one for $n = 3$. We add to the first column the last one multiplied by c_{2n-1}/κ and the $(n+1)$th column multiplied by $-(n-2)c_{n-1}c_{2n-1}/(\kappa^2 - n(n-2)c_{2n-1}^2)$ and the $(n-1)$th column multiplied by $c_{n-1}/(\kappa - n(n-2)c_{2n-1}^2/\kappa)$. We find that the first column of the determinant is now:

$$\begin{pmatrix} 2\kappa\rho + \frac{(2n-2)c_{2n-1}^2}{\kappa} + \frac{(n-2)c_{n-1}^2}{\kappa - n(n-2)c_{2n-1}^2/\kappa} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{76}$$

We deduce that:

$$\ln(R_c) = -\frac{(n-1)c_{2n-1}^2}{\kappa^2} - \frac{1}{2}\frac{(n-2)c_{n-1}^2}{\kappa^2 - n(n-2)c_{2n-1}^2}. \tag{77}$$

It can be checked that if $c_{2n-1} = 0$ and $c_{n-1} = m_1$ are introduced into (77), we find again the result (69) and also if $c_{n-1} = 0$ and $2n$ is replaced by n in (77).

These results can be applied to the cases of an approximate equilateral triangle and of an approximate square with a second order approximation.

Coefficients c_i are obtained from m_i by relation (72) with coefficients m_i obtained from the approximations at the second order of the triangle and square given in Appendix C.

For a triangle, with $n = 3$, coefficients c_i are given by $c_5 = 1/45$ and $c_2 = -47/135$, hence:

$$\ln(R_c) = -\frac{2}{2025}\frac{1}{\kappa^2} - \frac{2209}{36450}\frac{1}{\kappa^2 - \frac{1}{675}}. \tag{78}$$

For a square we have : $c_7 = 1/56, c_3 = -59/336$, hence:

$$\ln(R_c) = -\frac{3}{3136}\frac{1}{\kappa^2} - \frac{3481}{112896}\frac{1}{\kappa^2 - \frac{1}{392}}. \tag{79}$$

Eq. (78) is plotted in Fig. 2 (curve T2) and Eq. (79) is plotted in Fig. 3 (curve S2).

4.3.3. *The degree of* P *is 3*

This type of conformal mapping has also been considered by Ekneligoda and Zimmerman (2006, 2008). Writing c_1 instead of c_{n-1}, c_2 instead of c_{2n-1}, c_3 instead of c_{3n-1}, the value of R_c obtained along the previous lines is given by:

$$\begin{aligned} \ln(R_c) = & -\frac{1}{2}\frac{(3n-2)c_3^2}{\kappa^2} - \frac{1}{2} \\ & \times \frac{\kappa^2((n-2)c_1^2 + 2(n-1)c_2^2) - 2n(n-1)(n-2)(2c_3^2c_2^2 - 2c_3c_2^2c_1 + c_2^4 + c_1^2c_3^2)}{\kappa^4 - \kappa^2 n(2(2n-3)c_3^2 + (n-2)c_2^2) + 4n^2(n-1)(n-2)c_3^4}. \end{aligned} \tag{80}$$

Substituting $c_3 = 0$ into (80) gives the previous formula (77).

We focus now on the approximates of a square and of an equilateral triangle. Coefficients c_i are obtained from the mapping at the third order characterized by coefficients $m_i, i = 1, \ldots, 3$ given in Appendix C. Transforming again the rational fraction $\widetilde{f}$ by the polynomial quotient gives:

$$\begin{aligned} \widetilde{f} = & m_3.z^{3n-1} + (m_2 + (n-1)m_3m_1)z^{2n-1} + z^{n-1}(m_1^2m_3(n-1)^2 \\ & + m_1 + (n-1)m_1m_2 + (2n-1)m_3m_2) + f_1(z) \end{aligned} \tag{81}$$

that leads to the expression of coefficients c_i.

The results corresponding to the three approximations for a triangle are reported in Fig. 2 and the ones for a square in Fig. 3. These results are compared to the ones obtained numerically for polygons that are at the degenerate scale for Laplace equation, the sides of these polygons being recovered from Rumely (1989). The numerical results are obtained by the method described in Appendix B, based on the Boundary Element Method. The same method has been used by Vodička and Mantič (2008) and more recently by Chen et al. (2015a). It has been checked that the BEM leads to the same values of degenerate scales as the analytical ones when reproducing the contours corresponding exactly to the mappings. The numerical method reproduces also the case of the circle within a relative error of the order of 10^{-4}. It has also been checked with the same level of approximation that the degenerate scale for large values of κ tends to the degenerate scale for Laplace equation.

The results show that all approximations lead to similar values of degenerate scales, the accuracy increasing with the order of the solution. The largest discrepancy is attained when κ is near to 1, that corresponds to elastic incompressibility ($\nu = 0.5$). However, even in this case, the difference between numerical and analytical results remains small, within a few percent.

5. Case where $\widetilde{f}(z)$ has only one simple pole in C^{-}

In the previous section, there was only one pole at $z = 0$. Now, the case of one simple pole in C^{-} is studied.

5.1. *General solution*

We assume that function $\widetilde{f}$ is now given by:

$$\widetilde{f}(z) = a + \frac{b}{z - z_1} + f_0(z), \tag{82}$$

with $f_0(z)$ a holomorphic function on C^{-} and $z_1 \in C^{-}$. Then we need to find : $\phi = c/(z - 1/\overline{z_1})$. We have: $\widetilde{\phi} = -\overline{c}z_1 - \overline{c}z_1^2/(z - z_1)$.

We substitute these values of $\widetilde{f}$ and $\widetilde{\phi}$ into (42) and we write that the constant term and the coefficient of $1/(z - z_1)$ are equal to 0; we get a system of two equations where A and c are unknown:

$$\begin{cases} 2A\ln(R) - c\overline{z_1} = 0 \\ bA\frac{1}{z_1} + \kappa\overline{c}z_1^2 - c\frac{b\overline{z}_1^2}{(1 - z_1\overline{z}_1)^2} = 0. \end{cases} \tag{83}$$

After splitting the system (83) into real and imaginary parts, we can write the determinant of the system.

We focus now on the special case where z_1 and b are real. Then the determinant is:

$$D = \begin{vmatrix} 2\ln(R) & 0 & z_1 & 0 \\ 0 & 2\ln(R) & 0 & z_1 \\ \frac{b}{z_1} & 0 & \kappa z_1^2 - \frac{bz_1^2}{(1-z_1^2)^2} & 0 \\ 0 & \frac{b}{z_1} & 0 & -\kappa z_1^2 - \frac{bz_1^2}{(1-z_1^2)^2} \end{vmatrix}. \quad (84)$$

This determinant is the product of two determinants:

$$D = \begin{vmatrix} 2\ln(R) & z_1 \\ \frac{b}{z_1} & \kappa z_1^2 - \frac{bz_1^2}{(1-z_1^2)^2} \end{vmatrix} \cdot \begin{vmatrix} 2\kappa\ln(R) & z_1 \\ \frac{b}{z_1} & -\kappa z_1^2 - \frac{bz_1^2}{(1-z_1^2)^2} \end{vmatrix}. \quad (85)$$

The values of $\ln(R_c)$ are finally given by the values of R for which the determinant is null:

$$\ln(R_c) = \begin{cases} -\frac{b}{2z_1^2}\frac{1}{\kappa - b/(1-z_1^2)^2}; \\ \frac{b}{2z_1^2}\frac{1}{\kappa + b/(1-z_1^2)^2}. \end{cases} \quad (86)$$

5.2. Example of an ovoid

The example of the ovoid defined by $w(z) = z + m/(4z-1)$ has been given by Chen et al. (2009a). It corresponds to $y_1 = 1/4$, $z_1 = 4$. It is again an application of the previously described method. For this case, $\widetilde{f}$ is given by:

$$\widetilde{f}(z) = -m + \frac{900m}{(4m-225)(z-4)} + \omega_1(z) \quad (87)$$

where $\omega_1(z)$ is a holomorphic function in C^- vanishing at infinity.

The values of $\ln(R_c)$ are:

$$\ln(R_c) = \begin{cases} -\frac{b}{32\kappa}\left(1 + \frac{b/225}{\kappa - b/225}\right) = -\frac{b}{32}\frac{1}{\kappa - b/225}, \\ \frac{b}{32\kappa}\left(1 - \frac{b/225}{\kappa + b/225}\right) = \frac{b}{32}\frac{1}{\kappa + b/225}, \end{cases} \quad (88)$$

with

$$b = \frac{900m}{4m-225}. \quad (89)$$

That corresponds to the result obtained by Chen et al. (2009a).

6. Case where $w'(z)$ has one or several simple zeros on C

In Section 3, it was assumed that $w'(z)$ has no zero on C. We study now the case where $w'(z)$ has one or several simple zeros on C, and so $\widetilde{f}(z)$ has one or several simple poles on C.

6.1. Singularities of the curve Γ

Due to the zeros of $w'(z)$ on C, the curve Γ has one or several cusps.

Following Pommerenke (1992), Γ has a corner of opening $(\beta^- - \beta^+)$ at $z_0 = \exp i\upsilon_0$ if:

$$\arg\left(w(e^{it}) - w(e^{i\upsilon_0})\right) \to \begin{cases} \beta^+ & \text{as } t \to \upsilon_0^+ \\ \beta^- & \text{as } t \to \upsilon_0^- \end{cases}. \quad (90)$$

If $w'(z_0) = 0$ and $w''(z_0) \neq 0$, then using a series expansion of w in a neighborhood of z_0, we get $\beta^- - \beta^+ = 0\ (2\pi)$ and the curve Γ has a cusp (Pommerenke, 1992).

6.2. Modification of the way to find ψ

In (20) there are now some extra terms $b_i'/(z - z_i')$. The boundary equation can now be written

$$2\kappa\overline{A}\ln(R) + \kappa\overline{\phi}_0(z) - \left(\underbrace{\left(b_0 + \sum_{j=1}^{l_0} b_i z^i\right)}_{=P(z)} + \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{l_i}\frac{b_{i,j}}{(z-z_i)^j}}_{=F(z)} + \sum_{i=1}^{l_c}\frac{b_i'}{z - z_i'} + f_1(z)\right) \times \left(\phi_0'(z) + \frac{A}{z}\right) - \psi_0(z) = 0. \quad (91)$$

This case is solved by studying the following auxiliary problem to find the value of R, $A \neq 0$ ϕ_1 and ψ_1 by the method described in Section 3:

$$2\kappa\overline{A}\ln(R) + \kappa\overline{\phi}_1(z) - \left(\underbrace{\left(b_0 + \sum_{j=1}^{l_0} b_i z^i\right)}_{=P(z)} + \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{l_i}\frac{b_{i,j}}{(z-z_i)^j}}_{=F(z)} + f_1(z)\right)\left(\phi_1'(z) + \frac{A}{z}\right) - \psi_1(z) = 0. \quad (92)$$

When the problem corresponding to the boundary condition (92) is solved, the problem corresponding to the case with simple poles in C is solved by taking $\phi_0 = \phi_1$ and

$$\psi_0 = \psi_1 - \left(\sum_{i=1}^{l_c}\frac{b_i'}{z - z_i'}\right)\left(\phi_1'(z) + \frac{A}{z}\right). \quad (93)$$

Introducing ϕ_0 and ψ_0 into (91), it can, be checked that these functions constitute a solution of (91).

In this equation, we still assume that ϕ_1 is holomorphic in C^-, continuous in $C^- \cup C$ and tends to 0 as z tends to ∞. It must be noticed that ψ_1 has discontinuities in C. Nevertheless, the displacement is continuous in $C^- \cup C$.

Finally, *the only terms to consider in the partial fraction expansion of $\widetilde{f}(z)$ are the polynomial part and the partial fractions having their pole in C^-. The partial fractions with their poles on C can be disregarded.*

As a first consequence, this result allows to extend the results from hypotrochoids to hypocycloids.

6.3. Case of a segment of length 4

The conformal mapping is:

$$w(z) = z + 1/z; \quad (94)$$

hence:

$$\widetilde{f}(z) = z + \frac{1}{z-1} + \frac{1}{z+1}. \quad (95)$$

Due to the result of the previous subsection, both poles are on C and we have to consider only the polynomial term of $\widetilde{f}$. Then we have $b_1 = c_1 + id_1 = 1$.

The determinant (59) is reduced to the following 2×2 determinant:

$$\begin{vmatrix} 2\kappa\ln(R) - 1 & 0 \\ 0 & -2\kappa\ln(R) - 1 \end{vmatrix}. \quad (96)$$

We find the two critical scales, $R = \exp(-1/(2\kappa))$; $\exp(1/(2\kappa))$, that correspond to a result of Vodička and Mantič (2004).

6.4. Case of a circular arc

We can refer to Pommerenke (1992) for the definition of this conformal mapping:

$$w(z) = \frac{z(az+1)}{a(z+a)}. \quad (97)$$

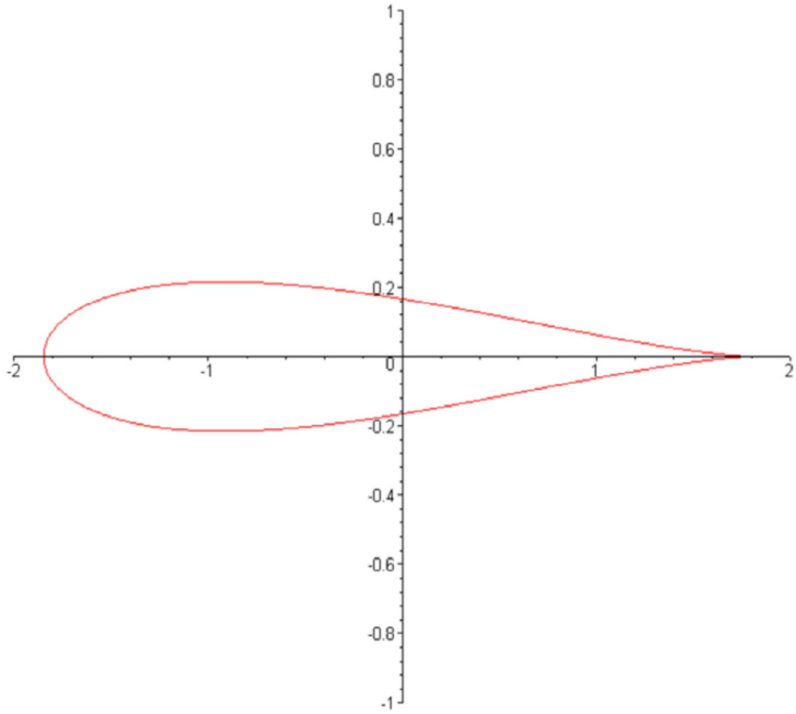

Fig. 4. Symmetrical Joukowski profile with $p = 0.1$.

The image of C by such a mapping is a circular arc $\{e^{i\theta}/a, -\alpha \le \theta \le \alpha\}$, with $a = \sin(\alpha/2)$. The function $\tilde{f}(z)$ that can be deduced from w is given by:

$$\tilde{f}(z) = \frac{(a^2-1)^2}{a^2\left(z+\frac{1}{a}\right)} + \frac{a^2}{z} + \frac{\sin(\alpha) - i\cos(\alpha)}{2\tan(\alpha/2)(z-(-\sin(\alpha/2) - i\cos(\alpha/2)))} + \frac{\sin(\alpha) + i\cos(\alpha)}{2\tan(\alpha/2)(z-(-\sin(\alpha/2) + i\cos(\alpha/2)))}. \tag{98}$$

This function has one pole $-1/a \in C^-$, one pole $0 \in C^+$ and two poles $-\sin(\alpha/2) \pm i\cos(\alpha/2) \in C$. Substituting $z_1 = -1/a$ and $b = (a^2-1)^2/a^2$ into (86) leads to the values of $\ln(R_c)$:

$$\ln(R_c) = \begin{cases} -\frac{1}{2}\frac{\cos^4(\alpha/2)}{\kappa - \sin^2(\alpha/2)}; \\ \frac{1}{2}\frac{\cos^4(\alpha/2)}{\kappa + \sin^2(\alpha/2)}. \end{cases} \tag{99}$$

For small values of α, the arc of circle tends to a segment of length 4, the values of R_c tend to $\exp(-1/2\kappa)$, $\exp(1/2\kappa)$ found in the previous section and for $\alpha \to \pi$ we find 1 as the limit value of (R_c), that is the double degenerate scale for a circle of radius 1 (Chen et al., 2002).

These closed form values can be compared to the numerical results of Chen et al. (2010). For instance, the critical values of the radius for $\kappa = 1.8$ and $\alpha = \pi/12$ are (5.84613, 10.0016) in Chen et al. (2010) and (5.85763, 10.0194) using (99), proving a very good agreement between both calculations.

6.5. Example of a symmetrical Joukowski profile

The exterior mapping is given by Ivanov and Trubetskov (1995) (see Fig. 4):

$$w(z) = \frac{1}{p+1}\left(t + \frac{1}{t}\right) \text{ with } t = z(p+1) - p;\ p > 0. \tag{100}$$

We have:

$$\tilde{f}(z) = -\frac{p^2+1}{(p+1)p} + \frac{1}{p^2+2p+1}\frac{1}{z-1} - \frac{5p^2+2p+1}{3p^4+4p^3-2p^2-4p-1}\frac{1}{z-\frac{(p-1)}{(p+1)}} + \frac{p^2}{p^2-1}\frac{1}{z} - \frac{4p^2+4p+1}{p^2(3p^2+4p+1)}\frac{1}{z-\frac{p+1}{p}}. \tag{101}$$

The poles inside and over the circle do not contribute to the solution. As a consequence, the only term that we have to consider is the last one. We apply (86) with $z_1 = (p+1)/p$ and $b = \frac{4p^2+4p+1}{p^2(3p^2+4p+1)}$. After simplification, we get:

$$\ln(R_c) = \begin{cases} -\frac{1}{2}\frac{(2p+1)^2}{((3p^2+4p+1)\kappa - p^2)(p+1)^2}; \\ \frac{1}{2}\frac{(2p+1)^2}{((3p^2+4p+1)\kappa + p^2)(p+1)^2}. \end{cases} \tag{102}$$

It can be checked that the limits of $\ln(R_c)$ are $\pm 1/2\kappa$ when $p \to 0$ and 0 when $p \to \infty$ as it should, from the previous section.

6.6. Example with two simple poles in C^-

We consider $w(z) = z - (9/8)(1/(z^2-1))$ (Fig. 5).

We have:

$$\tilde{f}(z) = \frac{8}{9} + \frac{25}{24}\frac{1}{z-2} - \frac{225}{184}\frac{1}{z+2} + \frac{33}{104}\frac{1}{z+1} + f_1(z). \tag{103}$$

Once again, the poles on the circle can be disregarded and we search for ϕ_0 with two unknown variables:

$$\phi_0 = \frac{b_{1,1}}{z-1/2} + \frac{b_{1,2}}{z+1/2}. \tag{104}$$

Using the procedure described in Section 3.4 leads to:

$$\ln(R_c) = \begin{cases} \frac{225}{64}\frac{-17+8\kappa}{-17+25\kappa+1242\kappa^2} \\ \frac{-225}{64}\frac{17+8\kappa}{-17-25\kappa+1242\kappa^2} \end{cases}. \tag{105}$$

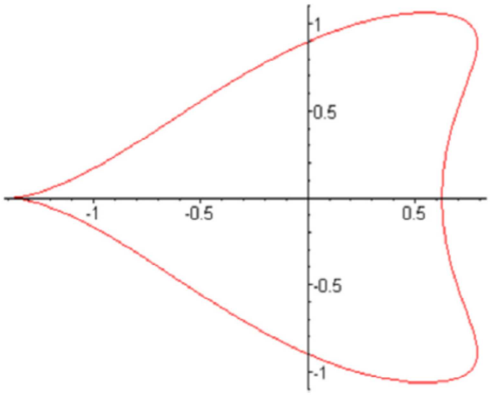

Fig. 5. Case of a curve with two poles in C^- and a zero of w' in C.

6.7. *Case of a mapping using a cyclotomic polynomial*

We consider a cyclotomic polynomial F_n of degree f_n (Lang, 1984, e.g.). Then we define $w(z) = \int (F_n(z)/z^{f_n})$. To avoid a logarithmic term in $w(z)$, it is necessary that the coefficient of z^{f_n-1} be equal to zero. That is true iff n is divisible by the square of an integer (Warusfel, 1971).

For example, the 9th cyclotomic polynomial is: $z^6 + z^3 + 1$, and we have $w(z) = z - 1/2z^2 - 1/5z^5$ (Fig. 6). We can now evaluate $\widetilde{w}(1/z)/w'(z)$:

$$\widetilde{f}(z) = -\frac{1}{5}z^5 - \frac{3}{10}z^2 + \frac{3z^2(5z^3+1)}{10(z^6+z^3+1)}. \tag{106}$$

Disregarding again the poles on C and applying (77) for $n = 2$, we find:

$$\ln(R_c) = -\frac{1}{200}\frac{625\kappa^2 - 48}{\kappa^2(25\kappa^2 - 3)}. \tag{107}$$

7. Degenerate scales obtained from the logarithmic capacity

7.1. *Sets of aligned segments*

In this section, a third method is introduced. This method allows us to find the degenerate scales for elasticity from the degenerate scale for Laplace's equation related to the same contour, itself being obtained from the logarithmic capacity.

7.1.1. *Relation between degenerate scales in plane elasticity and Laplace's problem for sets of aligned segments*

The following proposition specifies a first case where certain degenerate scales for elasticity can be deduced directly from the ones for Laplace's equation

Proposition. *If K is a set of aligned closed segments, then the two degenerate scales for plane elasticity (with the operator defined by complex potentials as in Section 2) of K are $\rho_1 = \rho_0 \exp(1/2\kappa)$ and $\rho_2 = \rho_0 \exp(-1/2\kappa)$, with ρ_0 being the degenerate scale for the Laplace's operator.*

Proof. We can assume that K is on the real axis and that its degenerate scale for the Laplace's operator is R_0 with the logarithmic capacity C_K equal to $1/R_0$. Then, there is a unique equilibrium measure γ such that (Landkof, 1972, e.g.):

$$\int_K \mathrm{d}\gamma(x) = -\frac{1}{\ln C_K}; \tag{108}$$

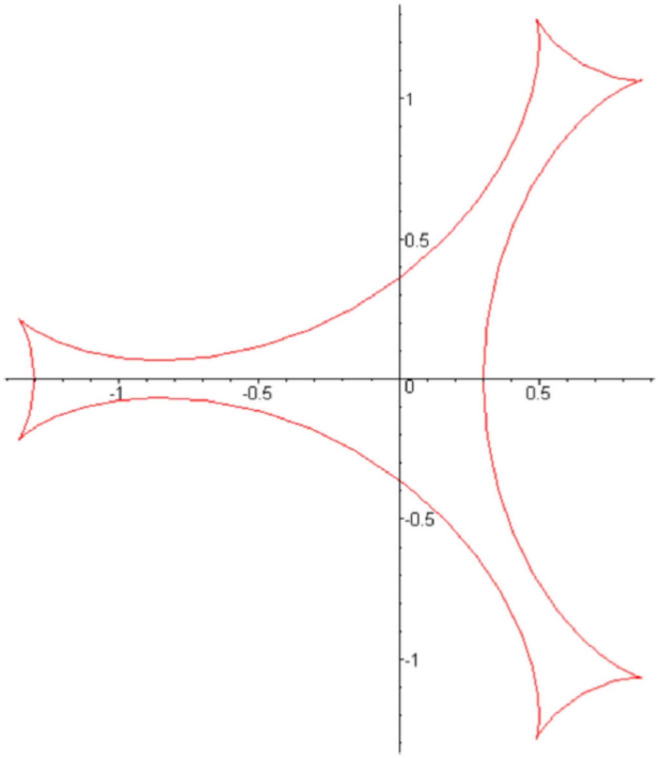

Fig. 6. Contour with 6 cusps corresponding to the 9th cyclotomic polynomial.

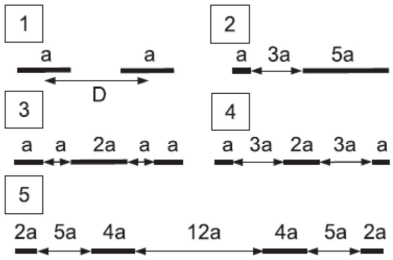

Fig. 7. Some cases of unions of segments.

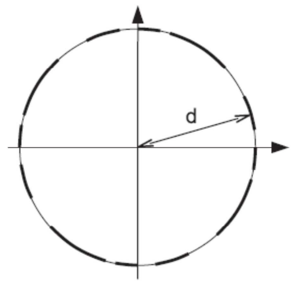

Fig. 8. Example of a set of arcs of circle with n-axis symmetry ($n = 4$).

$$U_\gamma = \int_K \ln \frac{1}{|x-y|} \mathrm{d}\gamma(x) = 1 \text{ for approximately all y} \in K. \tag{109}$$

In fact, the above equality holds $\forall y \in K$, as all the points satisfy the Wiener regularity criterion (Tsuji, 1959). In order to consider the image K_R of K by the scaling of ratio R, we introduce the following potential ϕ_R and we evaluate its derivative:

$$\phi_R(z) = A\left(\int_K \ln\left(\frac{z}{R} - z'\right) \mathrm{d}\gamma(z') - \frac{\ln R}{\ln C_K}\right) \; z \in K_R^-; \tag{110}$$

$$\phi_R'(z) = A \int_K \frac{1}{R} \frac{1}{\left(\frac{z}{R} - z'\right)} \mathrm{d}\gamma(z'). \tag{111}$$

We define also the potential ψ_R:

$$\psi_R = -\kappa \frac{\overline{A}}{A} \phi_R - z\phi_R' - \frac{A}{\ln C_K}. \tag{112}$$

When $z \to \infty$, we have $\phi_R = -A \ln z / \ln C_K + o(1)$, $z\phi_R' \to -A/\ln C_K$ and we can check that $\psi_R = \kappa \overline{A} \ln(z)/\ln(C_K) + o(1)$ when $z \to \infty$.

The value of the displacement on K_R for these potentials (ϕ_R, ψ_R) can be obtained using (6) with z instead of ζ ; as K_R is included in the real line, we have $\overline{z} = z$ and we get

$$\kappa \overline{\phi_R} - \overline{z}\overline{\phi_R'} - \overline{\psi_R} = 2\kappa \overline{A} \mathrm{Re}\left(\frac{\phi_R}{A}\right) + \frac{A}{\ln C_K}. \tag{113}$$

For $z \in K_R$, $z/R \in K$ and using (108) and (109), we get:

$$\begin{aligned} \mathrm{Re}\left(\frac{\phi_R}{A}\right) &= \mathrm{Re}\left(\int_K \ln\left(\frac{z}{R} - z'\right) \mathrm{d}\gamma(z')\right) - \ln R/\ln C_K\Big) \\ &= -\left(1 + \frac{\ln R}{\ln C_K}\right). \end{aligned} \tag{114}$$

Substituting (114) into (113), we conclude that the displacement is null on K_R if:

$$2\kappa\overline{A}\left(1 + \frac{\ln R}{\ln C_K}\right) = \frac{A}{\ln C_K}. \tag{115}$$

Using $C_K = 1/R_0$, the above equation has non zero solutions only for the following degenerate scales:

$$R_1 = R_0 \exp(1/2\kappa) \; ; \; R_2 = R_0 \exp(-1/2\kappa). \tag{116}$$

□

7.1.2. *Examples of applications*

Case of two equal segments. We use the results of Rumely (1989) which gives the logarithmic capacities for different sets, notably the unions of segments shown in Fig. 7. For two equal segments of length a with a distance D between their centers (case 1 of Fig. 7), the logarithmic capacity is equal to : $\sqrt{Da}/2$. We deduce the two degenerate scales:

$$R_1 = \frac{2}{\sqrt{aD}} \exp(1/2\kappa) \; ; \; R_2 = \frac{2}{\sqrt{aD}} \exp(-1/2\kappa). \tag{117}$$

This case has been studied by two authors Chen (2015); Vodička (2013) and they found using asymptotic methods the same solution as above for large values of D; they both noticed that this solution is accurate even if the condition $D \gg a$ is not fulfilled. This is now fully proved.

Some cases with unequal segments. We derive the degenerate scales for the cases 2, 3 and 4 of Fig. 7 from the values of their logarithmic capacities that are given in Rumely (1989):

$$\text{Case 2} : R_1 = 3^{-1/3} a^{-1} \mathrm{e}^{1/2\kappa}, R_2 = 3^{-1/3} a^{-1} \mathrm{e}^{-1/2\kappa}; \tag{118}$$

$$\text{Case 3} : R_1 = 3^{-1/6} a^{-1} \mathrm{e}^{1/2\kappa}, R_2 = 3^{-1/6} a^{-1} \mathrm{e}^{-1/2\kappa}; \tag{119}$$

$$\text{Case 4} : R_1 = 10^{-1/3} a^{-1} \mathrm{e}^{1/2\kappa}, R_2 = 10^{-1/3} a^{-1} \mathrm{e}^{-1/2\kappa}. \tag{120}$$

The last example with four segments is built using the decomposition of one number into two different sums of two squares; here, we use $6^2 + 17^2 = 10^2 + 15^2$.

Setting the origin at the middle of the $12a$ gap and applying z^2 we get two segments $[36a^2, 100a^2]$, $[225a^2, 289a^2]$. Translating these two segments of $(325/2)a^2$ and applying z^2 we get one segment $[(125/2)^2 a^4, (253/2)^2 a^4]$. Finally the initial set is the inverse image of this last segment by the polynomial $(z^2 - 325a^2/2)^2$. The logarithmic capacity of the inverse image of a compact subset E by a monic complex polynomial of degree n is equal to the n^{th} root of the logarithmic capacity of E (Rumely, 1989, e.g.). The logarithmic capacity of a segment of length L is $L/4$. We deduce that the logarithmic capacity of the initial 4 segments set is $((253/2)^2 - (125/2)^2)a^4/4)^{1/4}$ and finally the degenerate scales for case 5 are:

$$\text{Case 5} : R_1 = 2^{-1} 3^{-3/4} 7^{-1/4} a^{-1} \mathrm{e}^{1/2\kappa}, \tag{121}$$

$$R_2 = 2^{-1} 3^{-3/4} 7^{-1/4} a^{-1} \mathrm{e}^{-1/2\kappa}. \tag{122}$$

7.2. *Sets of arcs of a circle with* n*-axis symmetry*

7.2.1. *Relation between degenerate sales in plane elasticity and in Laplace's problem for sets of arcs with* n*-axis symmetry*

We consider a set K of arcs of a circle of radius d with a n-axis symmetry (Fig. 8).

The fundamental result on the degenerate scales of K is defined in the following proposition:

Proposition. *For a set of arcs of a circle with n-axis symmetry, the degenerate scale for plane elasticity is equal to the degenerate scale for the Laplace operator.*

Proof. We use the same approach as for the sets of segments on a line. We assume that the degenerate scale for the Laplace operator is

$R_0 \neq 1$. So we consider the corresponding $\gamma(x)$ satisfying (108) and (109). We define ϕ_R as in (110) and ψ_R as follows:

$$\psi_R(z) = -\kappa\frac{\overline{A}}{A}\phi_R(z) - \frac{d^2}{z}\phi_R'(z). \tag{123}$$

We have to check that $\phi_R'(z)/z$ tends to zero when $z \to \infty$ and is a holomorphic function in $C\backslash K_R$.

The first point is a consequence of the property of ϕ_R': $\phi_R'(z) = O(1/z)$, $z \to \infty$. For the last point, we have to check that $\phi_R'(z)/z$ has a limit when $z \to 0$. In a neighborhood of 0, $\phi_R(z)$ has a series expansion $a_0 + a_n z^n + a_{2n} z^{2n} + \cdots$ due to the n-axis symmetry of the problem and so $\phi_R'(z)/z = n a_n z^{n-2} + 2n a_{2n} z^{2n-2} + \cdots$. Then $\phi'(z)/z$ has a limit of 0 if $n \geq 2$ and the singularity at 0 is removable.

We write the condition of no displacement on K_R, using the fact that $\overline{z}$ equals $-d^2r^2/z$ if $z \in K_R$:

$$\kappa\overline{\phi_R} - \overline{z}\phi_R' - \psi_K = 2\kappa\overline{A}\mathrm{Re}(\frac{\phi_R}{A}) = -2\kappa\overline{A}\left(1 + \frac{\ln R}{\ln C_K}\right). \tag{124}$$

If $\ln R = -\ln C_K$, we can choose $A = 1$ as well as $A = i$; so we can build two solutions (ϕ_R, ψ_R) that lead to two displacements fields that are independent. The dimension of the corresponding real eigenspace is 2, and we conclude that there is only one degenerate scale (Vodička and Mantič, 2004, Theorem 1):

$$R_1 = R_2 = R_0 = \frac{1}{C_K}. \tag{125}$$

We remind that for a n-axis symmetric set with $n \geq 3$, there is only one degenerate scale Vodička and Mantič (2004). In the special case that was considered, the condition $n \geq 2$ appears to be sufficient. □

7.2.2. Case of n equal arcs equally spaced

We can find the degenerate scale of n equal arcs equally spaced by using the properties of the inverse image of a compact set by a polynomial (Ransford, 1995, e.g.). The logarithmic capacity of an arc of radius d^n of center 0 and angle α is $d^n \sin(\alpha/4)$. By the inverse of $g(z) = z^n$, this arc is transformed into n arcs of angle α/n and radius d. Its logarithmic capacity is $(d^n \sin(\alpha/4))^{1/n}$. Finally the logarithmic capacity of n equally spaced arcs of radius d and angle β is: $d(\sin(n\beta/4))^{1/n}$. We finally conclude that the degenerate scales for plane elasticity are equal to:

$$R_1 = R_2 = R_0 = \frac{1}{d\sin(n\beta/4)^{1/n}} \quad n \geq 2. \tag{126}$$

It can be observed that if $n = 2$, $\beta = 1/d$ and $d \to \infty$, the limit of the two arcs of circle is constituted of two vertical segments of length 1 separated by a distance $D = 2d$. Then, the asymptotic values of the two degenerate scales are $2/D^{1/2}$, in accordance with the result of Vodička (2013).

8. Conclusion

A methodology has been presented, that provides exact values of degenerate scales for plane elasticity in domains outside contours, these domains being described by a conformal mapping of the domain C^-exterior to a unit circle C. This methodology rests on precise properties of the conformal mappings and of the complex potentials used to obtain elastic solutions in plane domains. The degenerate scales correspond to elastic potentials compatible with a null displacement over the contour. This fundamental condition is expressed by using the elastic potentials in different forms to make the solution easier. The major part of the paper rests on the case when the conformal mapping is given by a rational fraction. The fundamental null displacement condition is transformed, leading to a finite homogeneous linear system. The degenerate scales are two scalars that cancel out the determinant of this homogeneous system. Writing the linear system rests on the partial fraction expansion of the function defining the conformal mapping. The linear system is written using poles in C^- or at ∞. Several examples have been solved, depending of the properties of the mapping, including mappings such as $z + P(z^{-1})$, P being a polynomial, mapping having a simple pole in C^- and mappings having poles in C^- and on C. In this last case, it has been shown that the poles on C can be disregarded in the evaluation of the degenerate scales.

A last fully different method has been described for some subsets of a line and of a circle. In this case it has been possible to link the degenerate scales of the plane elasticity problem to the degenerate scales of the Laplace's problems. When an exact value of this last degenerate scale is known, the exact values of the degenerate scales related to elasticity can immediately be found. To our knowledge, these exact values are the first known for the degenerate scales in plane elasticity related to multi connected sets.

Appendix A. Some results on Cauchy integrals (Muskhelishvili, 1953).

If f is a holomorphic function in C^- continuous on C with no pole at infinity ($f(z) \to 0$ if $z \to \infty$):

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z} = -f(z) \quad \forall z \in C^-. \tag{A.1}$$

For a constant, we have:

$$\frac{1}{2\pi i}\int_C \frac{a\mathrm{d}\zeta}{\zeta - z} = 0 \quad \forall z \in C^-. \tag{A.2}$$

If f is a holomorphic function in C^- continuous on C with no pole at infinity ($f(z) \to 0$ if $z \to \infty$):

$$\frac{1}{2\pi i}\int_C \frac{\overline{f(\zeta)}\mathrm{d}\zeta}{\zeta - z} = 0 \quad \forall z \in C^-. \tag{A.3}$$

If f is a holomorphic function in C^+ continuous on C:

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z} = 0 \quad \forall z \in C^-. \tag{A.4}$$

Appendix B. Numerical computation of degenerate scales for plane elasticity.

The numerical method that is used to compute the degenerate scales follows one of the methods described in Vodička and Mantič (2008).

If the contour $\Gamma_d = \lambda_0\Gamma$ is at the degenerate scale, the following integral equation has non-trivial solutions:

$$\int_{\lambda_0\Gamma} U_{ij}p_j \mathrm{d}s = 0 \tag{B.1}$$

where U_{ij} is the Kelvin's tensor given by:

$$U_{ij} = -\kappa\delta_{ij}\ln(r) + \frac{(x_i - y_i)(x_j - y_j)}{r^2}. \tag{B.2}$$

The integral operator $\mathbf{U}_\Gamma$ is denoted by:

$$\mathbf{U}_\Gamma(\mathbf{p}) = \int_\Gamma \mathbf{U} : \mathbf{p}\mathrm{d}s \tag{B.3}$$

with

$$\mathbf{U} = -\kappa\ln(r)\mathbf{I} + \mathbf{V}. \tag{B.4}$$

Passing from Γ to $\lambda\Gamma$ leads to:

$$\mathbf{U}_{\lambda\Gamma}(\mathbf{p}) = \int_{\lambda\Gamma}\mathbf{U} : \mathbf{p}\mathrm{d}s = \lambda\left[\mathbf{U}_\Gamma(\mathbf{p}) - \kappa\ln(\lambda)\int_\Gamma \mathbf{p}\mathrm{d}s\right] \tag{B.5}$$

and therefore:

$$\mathbf{U}_\Gamma(\mathbf{p}) = \kappa\ln(\lambda_0)\int_\Gamma \mathbf{p}\mathrm{d}s. \tag{B.6}$$

The idea developed in Vodička and Mantič (2008) is to search for **p** related to a constant value of the second member of the integral equation over the contour. So, the following integral equations are solved for the basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$:

$$\mathbf{U}_\Gamma(\mathbf{p}_1) = \mathbf{e}_1, \tag{B.7}$$

$$\mathbf{U}_\Gamma(\mathbf{p}_2) = \mathbf{e}_2. \tag{B.8}$$

Next, **p** is sought in the form:

$$\mathbf{p} = q_1\mathbf{p}_1 + q_2\mathbf{p}_2, \tag{B.9}$$

where q_i are constant terms.

This leads to:

$$\mathbf{U}_\Gamma(\mathbf{p}) = q_1\mathbf{U}_\Gamma(\mathbf{p}_1) + q_2\mathbf{U}_\Gamma(\mathbf{p}_2) \tag{B.10}$$

$$= q_1\mathbf{e}_1 + q_2\mathbf{e}_2 \tag{B.11}$$

and therefore:

$$q_1\mathbf{e}_1 + q_2\mathbf{e}_2 = \kappa \ln(\lambda_0)\left[q_1\int_\Gamma \mathbf{p}_1 ds + q_2\int_\Gamma \mathbf{p}_2 ds\right]. \tag{B.12}$$

One denotes:

$$\mathbf{t}_1 = \int_\Gamma \mathbf{p}_1 ds, \tag{B.13}$$

$$\mathbf{t}_2 = \int_\Gamma \mathbf{p}_2 ds. \tag{B.14}$$

These vectors can be obtained numerically from the distributions of $\mathbf{p}_i$ over the contour and therefore:

$$q_1\mathbf{e}_1 + q_2\mathbf{e}_2 = \kappa \ln(\lambda_0)[q_1\mathbf{t}_1 + q_2\mathbf{t}_2]. \tag{B.15}$$

This leads to:

$$q_1 = \kappa \ln(\lambda_0)[q_1 t_{11} + q_2 t_{21}], \tag{B.16}$$

$$q_2 = \kappa \ln(\lambda_0)[q_1 t_{12} + q_2 t_{22}] \tag{B.17}$$

where t_{ij} are the components of $\mathbf{t}_i$ with consistent notations. This set of equations constitutes an homogeneous system for unknowns q_i that has a non-trivial solution only if its determinant is null.

We introduce m given by: $\kappa \ln(\lambda_0) = \frac{1}{m}$. The determinant of the system with unknown q_i is given by:

$$\begin{vmatrix} t_{11} - m & t_{21} \\ t_{12} & t_{22} - m \end{vmatrix} = 0. \tag{B.18}$$

This means that m is one eigenvalue of the 2×2 matrix:

$$M = \begin{pmatrix} t_{11} & t_{21} \\ t_{12} & t_{22} \end{pmatrix}. \tag{B.19}$$

Having obtained the eigenvalues, it is easy to obtain the related two values of λ_0 that are the degenerate scales.

This method is applied to obtain numerical values of degenerate scales, the integral equations being discretized classically using constant elements. The contours are discretized using 500 elements.

Appendix C. Mappings for the approximate squares and polygons.

The mapping of an equilateral triangle is given by Muskhelishvili (1953):

$$w(z) = -A\int_1^z (1+t^3)^{2/3}\frac{dt}{t^2} + const. \tag{C.1}$$

Expanding into a power series leads, for a mapping which is order of z at infinity, to:

$$w(z) = z - \frac{1}{3z^2} + \frac{1}{45z^5} - \frac{1}{162z^8} + \cdots \tag{C.2}$$

For the mapping of a square, one has:

$$w(z) = -A\int_1^z (1+t^4)^{1/2}\frac{dt}{t^2} + const. \tag{C.3}$$

and

$$w(z) = z - \frac{1}{6z^3} + \frac{1}{56z^7} - \frac{1}{176z^{11}} + \cdots \tag{C.4}$$

Both mappings are of the form:

$$w(z) = z + \frac{m_1}{z^{n-1}} + \frac{m_2}{z^{2n-1}} + \frac{m_3}{z^{3n-1}}. \tag{C.5}$$

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